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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

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On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 19 parts. The 19 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A plane-wave solution of the Dirac equation has the form

$$\psi(x) = u(\vec{p}, s)e^{-ipx},$$

where

$$u(\vec{p}, s) = \frac{\gamma^\mu p_\mu + m}{\sqrt{2m(m + \omega_p)}} u(0, s),$$

with $s = 1, 2$, and

$$u(0,1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u(0,2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

In this problem the momentum p satisfies $p^0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$.

1.1 Show that

$$\Lambda_+ u(\vec{p}, s) = u(\vec{p}, s),$$

where

$$\Lambda_+ = \frac{m + \not{p}}{2m}.$$

1.2 Show that the matrix P_{ab} defined by

$$P_{ab} = \sum_{s=1}^{2} u_a(\vec{p}, s)\bar{u}_b(\vec{p}, s),$$

satisfies

$$P = \Lambda_+ .$$

1.3 The vector $V^\mu(p, s, t)$ is given by

$$V^\mu(p, s, t) = \bar{u}(\vec{p}, s)\gamma^\mu u(\vec{p}, t) .$$

Calculate $V^\mu(p, s, t)^*$.

1.4 Show that

$$\sum_{s,t=1}^{2} V^\mu(p, s, t)^* V_\mu(p, s, t)$$

can be written as the trace of a product of 4×4 matrices.

1.5 Show that

$$\gamma^\mu\gamma_\mu = 4\mathbb{1}_4 , \quad \text{and that} \quad \gamma^\mu\gamma^\rho\gamma_\mu = -2\gamma^\rho .$$

1.6 Evaluate the trace obtained in (1.4).

PROBLEM 2

The Dirac equation for an electron in an electromagnetic field is given by

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi(x) = 0 .$$

Assume that A_μ corresponds to a radial Coulomb potential, i.e., only $A^0 \neq 0$, and A^0 depends only on $\vec{x}^2$. The orbital angular momentum operator is $\vec{L} = \vec{x} \times \vec{p}$. The operators $\vec{p} = (p^1, p^2, p^3)$ are represented by $p^k = -i\partial_k$ The spin angular momentum is given by $\vec{S} = \frac{1}{2}\gamma^5\gamma^0\vec{\gamma}$.

2.1 Write the Dirac equation in the form

$$i\frac{\partial}{\partial t}\psi(x) = H\psi(x) .$$

Determine the Hamiltonian H.

2.2 Show that $[\vec{L},\ A^0(x^2)] = 0$.

2.3 Evaluate $[\vec{L},\ \gamma^k p_k]$.

2.4 Show that $[\vec{L},\ H] = i\gamma^0(\vec{\gamma}\times\vec{p})$.

2.5 Show that $[\vec{S}, \gamma^0] = 0$.

2.6 Show that $[\vec{L}+\vec{S},\ H] = 0$.

PROBLEM 3

The Dirac field $\psi(x)$ satisfies equal-time anticommutation relations

$$\{\psi_a(x), \psi_b^\dagger(y)\}_{x^0=y^0} = \delta_{ab}\delta^3(\vec{x}-\vec{y})\,,$$

where $a, b = 1, \ldots, 4$ are spinor indices.

3.1 What is the definition of the time-ordered product $T(\psi(x)\bar{\psi}(y))$?

3.2 Show that

$$(i\gamma^\mu\partial_{x^\mu} - m)T(\psi(x)\bar{\psi}(y)) = i\delta^4(x-y)\,. \tag{3.2}$$

3.3 Given that $(\partial_x^2 + m^2)\Delta_{\mathrm{F}}(x-y) = -\delta^4(x-y)$, show that

$$T(\psi(x)\bar{\psi}(y)) = i(i\gamma^\mu\partial_{x^\mu} + m)\Delta_{\mathrm{F}}(x-y)$$

satisfies (3.2).

PROBLEM 4

We consider elastic scattering of two electrons, with incoming momenta p_1 and p_2. The outgoing electrons have momenta p_3 and p_4. According to the Feynman rules of quantum electrodynamics ingoing electrons are represented by $u(p,s)$, outgoing electrons by $\bar{u}(p,s)$. The Feynman rule for each vertex has a factor $-ie\gamma^\mu$, momentum and charge are conserved at each vertex, and an internal photon line is represented by

$$-ig_{\mu\nu}\,\frac{1}{q^2}\,,$$

where q is the photon momentum.

4.1 Draw the Feynman diagrams corresponding to the elastic scattering of two electrons at order e^2 in perturbation theory.

4.2 Write the amplitude $M(p_1, p_2, p_3, p_4)$ corresponding to these diagrams.

4.3 Express the photon momentum q in terms of the momenta of the electrons for each of the diagrams in (4.1).

4.4 What is sign of q^2 for each of the diagrams in (4.1)?